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George J. Fix

Max D. Gunzburger

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George J. Fix*

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ABSTRACT

Finite element methods are introduced for the approximate solution of periodic acoustic problems. A least squares technique is used for those problems which are governed by a first order system of partial differental equations while for second order equations, a Galerkin/ multigird technique is employed. In both cases, the solution process for the algebraic system resulting from discretization is iterative in character.

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* Department of Mathematics, Carnegie-Mellon University, Pittsburgh, PA 15213

† Department of Mathematics, University of Tennessee, Knoxville, TN 37916

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I. Introduction

The purpose of this work is to introduce new methods for the approximate solution of equations which govern the propagation of linear periodic waves through non-uniform media. Such phenomena occur in numerous branches of physics, e. g. see [1-5]. In many cases the equations which govern the propagation of waves form a first order system of hyperbolic equations

$$\frac{\partial \underline{\Phi}}{\partial t} + \sum_{j=1}^{n} A_j \frac{\partial \underline{\Phi}}{\partial x_j} + B\underline{\Phi} = \underline{f}(\underline{x})\exp\{i\omega t\} \tag{1.1}$$

for $x \in \Omega$, the latter being the domain of interest. The summation limit n is the number of space dimensions that the solution vector $\underline{\Phi}$ depends on, A_j and B are real valued coefficient matrices which may depend on $\underline{x}$ (but not on t) and $\underline{f}$ is a complex valued vector forcing function. If $\underline{\Phi}$ is assumed to be periodic in time with the same frequency as the forcing function, i.e.

$$\underline{\Phi} = \underline{\phi}(\underline{x})\exp\{i\omega t\} \quad , \tag{1.2}$$

then equation (1.1) reduces to the complex valued equation

$$\sum_{i=1}^{n} A_j \frac{\partial \underline{\phi}}{\partial x_j} + (B + i\omega I)\underline{\phi} = \underline{f} \tag{1.3}$$

The existence of periodic solution of equation (1.1) for general A_j, B and $\underline{f}$ is not immediately obvious, and of course, also depends on the boundary condition imposed on $\underline{\Phi}$. However, for a large class of problems

there is preponderant physical evidence that such solutions do exist. Furthermore this conclusion is also supported by some preliminary mathematical investigation [6], [7].

The coefficient matrices for the governing system (1.3) and the attendant well posed boundary conditions may drastically vary in appearance from one problem to another. Indeed, in some very simple instances, the system (1.3) may be reduced to a second order equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial P}{\partial x_j} \right) + \sum_{i=1}^{n} b_i \frac{\partial P}{\partial x_i} + c P = F \qquad (1.4)$$

where P may be one component of $\underline{\phi}$ or some other variable, and where the coefficients a_{ij}, b_i and c and the function F are in general complex valued. In this paper we will discuss numerical methods for the solution of systems of the type (1.3) and equations of the type (1.4).

There are four separate problems that may be posed in conjunction with equations such as (1.3) and (1.4). The particular problem type is determined by the extent of the domain Ω and the nature of the boundary conditions imposed on Γ, the boundary of Ω. We classify those problems for which the domain Ω is bounded in its extent as *interior problems*. In general it will be required that only one boundary conditions be specified on Γ. Of course, the form of the boundary condition may vary, usually in a piecewise manner, as one travels along Γ. Typically, for the system (1.3) one component or a linear combination of some of the components of $\underline{\Phi}$ is specified on Γ, i.e.

$$\underline{\mu} \cdot \Phi = g(\underline{x}) \qquad \text{for} \quad \underline{x} \in \Gamma \qquad (1.5)$$

where the function g and the component of the vector $\underline{\mu}$ are piecewise "smooth" function of $\underline{x} \in \Gamma$. For the second order equation (1.4), the boundary conditions would typically look like

$$\alpha P + \underline{\beta} \cdot \text{grad } P = G(\underline{x}) \quad \text{for} \quad \underline{x} \in \Gamma \tag{1.6}$$

where the functions G and α and the components of the vector $\underline{\beta}$ are also piecewise "smooth" functions of $\underline{x} \in \Gamma$.

The coefficients of the system (1.3) or the second order equation (1.4) are in general functions of ω. For a discrete set of values of ω non-trivial solutions of equations (1.3) and (1.5) [or equations (1.4) and (1.6)] with $\underline{f} = 0$ and $g = 0$ [or $F = 0$ and $G = 0$] exist. Those values of ω are called the eigenvalues of the homogeneous problem determined by the differential equation, the domain Ω, and the boundary condition specified on Γ. The corresponding non-trivial solutions are called eigenfunctions. Then in general, if ω is an eigenvalue of a specified homogeneous problem, the corresponding inhomogeneous problem may not have a solution, and even when it does, the solution will not be unique. This is, of course, the renowned Fredholm alternative [8]. Therefore, in specifying an interior problem, we tacitly assume that ω is not an eigenvalue. On the other hand, given a domain Ω, a differential equation or system, and boundary conditions imposed on Γ, it is of practical interest to determine those values of ω, i.e. the eigenvalues, for which non-trivial solutions to the homogeneous problem exist. This type of problem is usually called an *eigenvalue problem* and is a second problem that may be posed for equation of the type (1.3) or (1.4).

A third type of problem, namely *exterior problems*, are characterized by domains whose extent is infinite in every direction, i.e. Ω is exterior

to some bounded domain. Here boundary conditions are imposed on the bounded surface to which Ω is exterior of, as well as at infinity. It is well known [9], [10] that for the Helmholtz equation

$$\Delta P + k^2 P = 0, \tag{1.7}$$

unique solutions exist (for any k) only if P satisfies the radiation condition (in three-dimensions)

$$\lim_{r \to \infty} \left\{ r \left| \frac{\partial P}{\partial r} + i k P \right| \right\} = 0. \tag{1.8}$$

where r is the distance from the origin of the coordinate systems. It is often the case that as $r \to \infty$, the system (1.3) or the equation (1.4) may be combined, if not exactly, at least asymptotically, to yield the Helmholtz equation (1.7). Therefore the radiation condition (1.3) is again applicable, where for the system (1.3) the P derivative is replaced by an appropriate linear combination of the components of $\underline{\Phi}$.

The fourth type problem is one in which Ω is finite in its extent in some directions, and infinite in others. These problems arise in the propagation of waves in oceans, layered media, wave guides, etc. A study of the numerical solution of such problems has been effected by Fix and Marin [11] and will not be considered here.

The approximation solution of the linear wave problems described above have numerous inherent difficulties attached to them. Chief among these are the problem of resolution, of indefiniteness and, for the exterior problems, of infinite domains.

The *resolution problem* can be described as follows: for a fixed domain Ω, as the forcing frequency increases one can expect an increase in the number

of waves present in Ω, waves which have to be adequately described by the approximate solution. For solutions in terms of eigenfunction expansions or via fast Fourier transform techniques, the adequate resolution of waves will require, at moderate values of ω, the keeping of a prohibitive number of terms in the series. For the limited class of problems for which a free space Green's function, i.e. a fundamental solution of the differential equation, is known, the resolution problem can best be alleviated by transforming the given problem into an integral equation. This approach reduces the number of space dimensions by one, but is in general limited to problems with very simple mean flows. For finite difference or finite element discretizations, the resolution of waves can be accomplished only by choosing a fine enough grid. This restriction on the grids is one quite apart from questions of accuracy as determined by truncation error analyses.

Standard finite difference, finite element or spectral discretizations of (1.3) result in an *indefinite*, nonsymmetric linear system of algebraic equations. Furthermore, even in the self-adjoint case, i.e. $b_i = 0$, discretizations of (1.4) will also result in *indefinite* algebraic systems for large enough values of the forcing frequency ω. Certainly, in the case of finite difference and finite element methods, the algebraic systems in question will in general be sparse in the sense that they are banded and that the non-zero entries in any row of the coefficient matrix (of the algebraic system) is in general a constant independent of the number of unknowns, i.e. the grid size. In view of the resolution problem it would be advantageous to solve the algebraic system resulting from discretization by an iterative method since these methods would require computer storage proportional to the number of unknowns (for the problems under consideration here).

Unfortunately, the standard iterative methods, e.g. Gauss-Seidel, SOR, are not applicable to indefinite linear systems. Therefore, one is forced to use some form of Gauss elimination to solve the linear system. For the banded linear systems in question here, the storage requirements are roughly proportional, in two-dimensions, to $N^{3/2}$, where N is the number of unknowns.

In this work separate iterative techniques are described for solving systems of the type (1.3) and second order equations of the type (1.4). In the first case we circumvent the above problems by introducing a least squares discretization of the system (1.3). The resulting algebraic system is then symmetric and positive definite, thus enabling the use of standard iterative methods such as Gauss-Seidel or SOR. This least squares-method is discussed in section 3. For the second order equation (1.4), a standard Galerkin/finite element discretization is used, yielding perhaps an indefinite algebraic system. This system is solved in an iterative manner by using the multigrid technique of Nicolaides [12], [13], [14]. This technique is discussed in section 4. In both cases the overall methods are insensitive to the value of ω insofar as their iterative character is concerned.

For exterior problems the domain Ω is *infinite* in its extent. Once again, for those special problems for which a free space Green's function is known, the best approach to exterior problems is to convert them into integral equations, thus not only reducing the number of independent variables, but also rendering the lower dimensional domain finite [15]. For more complicated problems one turns to finite element or finite difference discretizations. Of course, one must now choose a finite computational domain. One apparent way of doing so is to map the infinite domain into a finite one. However, since the problems of interest here have wave-like solutions, such mappings would still result in the need of an infinite number

of grid points in order to resolve the infinite number of waves compressed into the finite mapped domain. The alternative is to truncate the infinite domain and then impose approximate radiation conditions on the boundary of this finite domain. This process is discussed briefly in the first appendix.

II. Model Problems

To fix ideas we will now present, in some detail, the derivation of the governing equations for the propagation of linear periodic waves through a lossless fluid which is itself in steady motion. For the sake of simplicity we will restrict ourselves to two space dimensions. Furthermore, we will present typical boundary conditions for an interior problem. The starting point in our analysis is those equations which govern the general motion of a lossless fluid, i.e. the Eulerian equation of motion

$$\frac{\partial p}{\partial t} + \underline{u}\cdot\text{grad } p + \gamma p \text{ div } \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\text{grad})\underline{u} + \frac{1}{\rho}\text{grad } p = 0 \qquad (2.1)$$

$$p\,\rho^{-\gamma} = \text{constant}$$

where ρ, p and $\underline{u}$ are the fluid density, pressure and velocity, respectively, and γ is the constant ratio of the specific heats. The last equation appearing in the system (2.1) is due to assuming that the fluid is perfect and the motion is isotropic. In the absence of any acoustic disturbance the fluid is assumed to be in steady motion. For such motion, we denote the dependent variables with a zero subscript.

The first central assumption of linear acoustics is that the acoustic disturbance in a small perturbation of the undisturbed steady flow, i.e.

$$p(x,y,t) = p_0(x,y) + p'(x,y,t) \tag{2.2}$$

with

$$|p'/p_0| \ll 1 \tag{2.3}$$

where the prime superscript denotes the acoustic variable. Similar relations hold for the other dependent variables. Substitution of equation (2.2) and the analogous expressions for ρ and $\underline{u}$ yields, upon neglecting terms that are quadratic in the acoustic variables, the system (1.1) with $n = 2$, $x_1 = x$, $x_2 = y$, $\underline{f} = 0$,

$$A_1 = \begin{pmatrix} u_0 & \gamma p_0 & 0 \\ 1/\rho_0 & u_0 & 0 \\ 0 & 0 & u_0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} v_0 & 0 & \gamma p_0 \\ 0 & v_0 & 0 \\ 1/\rho_0 & 0 & v_0 \end{pmatrix} \tag{2.4}$$

$$B = \begin{pmatrix} u_{0x} + v_{0y} & p_{0x} & p_{0y} \\ -p_{0x}/(\gamma\rho_0 p_0) & u_{0x} & u_{0y} \\ -p_{0y}/(\gamma\rho_0 p_0 & v_{0x} & v_{0y} \end{pmatrix} \quad \text{and} \quad \underline{\Phi} = \begin{pmatrix} p' \\ u' \\ v' \end{pmatrix}$$

where u and v are the components of u in the x and y direction, respectively. Since $\underline{f}$ vanishes, it is assumed that the acoustic perturbation is caused by periodic boundary disturbances. The boundary conditions will in general be linear and whenever they are not homogeneous, the inhomogeneity will be a periodic function of time, i.e. proportional to $\exp(i\omega t)$.

The second central assumption of acoustics is that the solution of the initial value problem with its attendant periodic boundary conditions will

result, perhaps after some time or in an asymptotic manner, in a periodic solution (in time) with the same frequency ω as the forcing term. This assumption enables us to substitute equation (1.2) into the system (1.1), yielding the system (1.3) as the governing system for our duct acoustics problem. We denote the components of $\underline{\phi}$ by the relation

$$\underline{\phi} = \begin{pmatrix} P \\ U \\ V \end{pmatrix} . \tag{2.5}$$

If the fluid is in uniform motion in the x-direction, i.e. $u_0 = u_\infty$, $v_0 = 0$, ρ_0 and p_0 constant, the system for P, U, and V simplifies to

$$\begin{aligned} i\omega P + u_\infty P_x + \gamma p_0[U_x+V_y] &= 0 \\ \rho_0(i\omega U + u_\infty U_x) + P_x &= 0 \\ \rho_0(i\omega V + u_\infty V_x) + P_y &= 0 \end{aligned} \tag{2.6}$$

which may be combined into the second order equation (1.4) with $a_{11} = (1-M^2)$, $a_{12} = a_{21} = 0$, $a_{22} = 1$, $b_1 = -2ikM$, $b_2 = 0$ and $c = k^2$, where $M = u_\infty/a_0$, $k^2 = \omega^2/a_0^2$ = constant and $a_0^2 = \gamma p_0/\rho_0$ is the square of the speed of sound of the undisturbed fluid. Of course, if $u_\infty = 0$, the system (2.6) reduces to the Helmholtz equation (1.7). There are other situations in which the system (1.3) for $\underline{\phi}$ can be combined into a single equation for either P or possibly for a velocity potential function, e.g. see [3]. However, in general settings, the system (1.3) governing $\underline{\phi}$ cannot be combined into a single equation.

If the mean flow is everywhere subsonic, the system (1.3) with coefficients given by equation (2.4) is of elliptic type, as is the second order equation (1.4). In either case we are required to specify one boundary condition everywhere on the boundary of the domain . For an interior problem this boundary condition would take the form of equation (1.5) or equation (1.6). In particular, in many applications, the boundary conditions take the specific form

$$P = g(\underline{x}) \quad \text{for} \quad \underline{x} \in \Gamma_1 \tag{2.7}$$

and

$$\underline{n}\cdot\underline{U} = z(\underline{x})P \quad \text{for} \quad \underline{x} \in \Gamma_2 \tag{2.8}$$

or

$$\underline{n}\cdot\text{grad } P = i\omega\rho_0 z(\underline{x})P \quad \text{for} \quad \underline{x} \in \Gamma_2 \tag{2.9}$$

where $\Gamma_1 \cup \Gamma_2 = \Gamma$, the boundary of Ω, the complex function g represents a pressure disturbance entering Ω through Γ_1, the complex function z is the (possibly variable) impedence of the boundary Γ_2, and $\underline{n}$ is the unit normal to Γ_2. The combination of boundary conditions (2.7) and (2.8) are used with the system (1.3) while the combination of equations (2.7) an (2.9) are used with equation (1.4) or (1.7).

III. The Least Squares Finite Element Method

In this section we present a formal description of the least squares method for the approximate solution of ystems of the type (1.3). In order to simplify our discussion, let us formally write the system (1.3) as

$$L\,\underline{\phi} = \underline{f} \quad \text{in} \quad \Omega$$

and the boundary conditions as

$$B\,\underline{\phi} = g \quad \text{on} \quad \Gamma.$$

Here L is a first order linear differential operator, and B is a $1 \times m$ matrix, m being the dimension of $\underline{\phi}$, i.e. the number of unknowns. The least squares scheme is simply to minimize the functional

$$\int_{\Omega} |L\underline{\phi} - \underline{f}|^2 + \sigma \int_{\Gamma} |B\underline{\phi} - g|^2 \tag{3.1}$$

over a finite dimensional space S^h of vector valued functions. The weight σ in essence "balances" the interior and boundary integrals, and is essential for achieving the best possible accuracy [16]. Furthermore, σ will depend on the dimension of S^h. A mathematical analysis of this scheme can be found in Fix, et al. [17], [18].

Although finite element spaces using polynomials of any order could be used, to fix ideas we shall restrict our attention to linear elements. We let h parametrize the subdivision of Ω into subregions, e.g. let h be the maximum linear dimension of any subregion. Then arbitrary subdivision of the region Ω will not result in optimal accurate approximations. (Here by optimal accuracy is meant the highest possible power of h obtainable for the error in approximating the solution by elements in S^h.) It was shown by Fix, et al. [18] that linear elements in directional triangles (see Figure 1a) and bilinear elements in quadrilaterals (see Figure 1b) do not result in optimally accurate approximations. On the other hand linear elements in the "criss-cross" grid of Figure 1c do yield optimally accurate approximations. Indeed, using the "criss-cross" grid, the predicted accuracy is $0(h^2)$ for each of the components of $\underline{\phi}$. Therefore we assume that Ω can be divided into quadrilateral subregions and then, by drawing diagonals, each quadrilateral can be divided into four

triangles. This construction yields a "criss-cross" type grid. (We do not entertain here the well documented [19], [20] methods of treating curved boundaries.)

To continue our description of the method, we first let $\underline{z}_i = (x_i, y_i)$ denote the nodes as shown in Figure 1c. For each node we have a shape function ψ_i^h which is one at $\underline{z}_i$ and zero at all other nodes. We write

$$\underline{\phi}^h(x,y) = \sum_{i=1}^{M} \underline{\phi}_i^h \psi_i(x,y) \tag{3.2}$$

where M is the number of nodes and $\underline{\phi}_i^h$ is the vector of nodal values of the approximation. We substitute equation (3.2) for $\underline{\phi}$ in the functional (3.1) and then minimize over the space S^h. In this instance S^h has for a basis the set

$$\underline{e}_j \, \psi_i(x,y) \qquad i=1,\cdots,M \qquad j=1,\cdots,m$$

where $\underline{e}_j$ is the j-th unit vector of dimension m.

The above procedure produces the algebraic system

$$C \, \underline{\phi}_i^h = \underline{c} \tag{3.3}$$

Any details concerning the formation of the system (3.3) which have been omitted in the above description are common to all finite element methods, and for them the reader is referred to any of the many finite element methods texts now available, e.g. [19] or [20]. The one exception is that the weight σ appearing in the functional (3.1) should equal C/h, C a constant [16].

The matrix C is an $N \times N$ symmetric, positive definite, banded matrix. In addition, the number of non-zero entries in any row is independent of h.

If, for instance, Ω is the unit square and if there are n nodes along any given row [so that the mesh spacing h is $O(1/n)$], then the number of unknowns is

$$N = mn^2 + O(n)$$

and the half bandwidth is $mn + O(1)$.

The highly desirable matrix properties of symmetry, positive definiteness and sparsity are of course a crucial aspect of the least squares approach and the key to the method's insensitivity to the value of ω. Insofar as the applicability of iterative methods such as Gauss-Seidel or SOR for the solution of the system (3.3) is concerned, the first two properties are necessary. The third property, of course, makes the use of iterative methods desirable, especially in view of the resolution problem discussed in section 1. Furthermore these properties would not be obtained if one used a Galerkin or finite difference approximation to the system (1.3). Indeed, both these approaches would yield a system

$$D \underline{\phi}_1^h = \underline{d} \tag{3.4}$$

where D is an indefinite matrix of roughly the same size and typically the same structure as C.

Numerical Examples

The first example problem is one whose exact solution is known and is presented in order to illustrate the accuracy of the least squares method. Specificially, we consider the system (2.6) with $u_\infty = 0$. By introducing the nondimensionalizations

$$\hat{p} = P/\rho_0 a_0^2 \qquad \hat{u} = U/a_0 \quad \text{and} \quad \hat{v} = V/a_0$$

we then have that

$$\begin{aligned} \hat{u}_x + \hat{v}_y + ik\hat{p} &= 0 \\ \hat{p}_x + ik\hat{u} &= 0 \\ \hat{p}_y + ik\hat{v} &= 0 \end{aligned} \tag{3.6}$$

where again $k = \omega/a_0$. It is clear that p and therefore u and v satisfy the Helmholtz equation

$$\Delta\hat{\underline{\Phi}} + k^2\hat{\underline{\Phi}} = 0 \tag{3.7}$$

In the numerical experiments we take Ω to be the unit box $\{0<x<1, 0<y<1\}$ and use for boundary conditions the relations

$$\hat{v}(x,0) = \hat{v}(x,1) = 0 \quad \text{for} \quad 0 \leq x \leq 1$$

and (3.8)

$$\hat{u}(1,y) = 0, \quad \hat{p}(0,y) = \cos(\pi y) \quad \text{for } 0 \leq y \leq 1$$

With these boundary conditions the exact solution of the system (3.6) is given by

$$\hat{p}(x,y) = \cos(\pi y)\cosh[\mu(1-x)]/\cosh\mu \quad \text{if} \quad k^2 < \pi^2$$

and

$$\hat{p}(x,y) = \cos(\pi y)\cos[\mu(1-x)]/\cos\mu \quad \text{if} \quad k^2 > \pi^2$$

where $\mu = |k^2-\pi^2|^{\frac{1}{2}}$. The exact solution for $\hat{u}$ and $\hat{v}$ can easily be derived from these formulas and the system (3.6). Finally, for this problem, the functional (3.1) to be minimized is given by

$$\int_0^1\int_0^1 dxdy\{|\hat{u}_x+\hat{v}_y+ik\hat{p}|^2 + |\hat{p}_x+ik\hat{u}|^2 + |\hat{p}_y+ik\hat{v}|^2\}$$

$$+\frac{C}{h}\int_0^1 dx\{|\hat{v}(x,0)|^2 + |\hat{v}(x,1)|^2\} + \frac{C}{h}\int_0^1 dy\{|\hat{u}(1,y)|^2 + |\hat{p}(0,y)-\cos(\pi y)|^2\}$$

We denote by $\hat{p}^h$, $\hat{u}^h$ and $\hat{v}^h$ the approximate least squares solution of the system (3.6) and the boundary conditions (3.8). The L_2 error of the approximation to $\hat{p}$ is defined to be

$$e_p = \|\hat{p}-\hat{p}^h\| = \{\int_\Omega |\hat{p}-\hat{p}^h|^2\}^{\frac{1}{2}}.$$

Analogous expression hold for e_u and e_v. We also define the error

$$E_p = \max_{j=1,m} |\hat{p}(z_j)-\hat{p}^h(z_j)|$$

which is the maximum absolute error at the nodes. Figures 2 and 3 display the L_2 errors in the least squares approximation to u, v and p as functions of h for the problem defined by equation (3.6) and (3.8). Also shown in these two figures is E_p as a function of h. These errors were computed for approximation based on the linear "criss-cross" triangles described earlier in this section. Figure 2 is for $k = \omega/a_0 = 1$ and Figure 3 is for $k = 7/4$. Note that the second value of k is greater than $\pi/2$ so that for that k the Helmholtz equation (3.7), with the boundary conditions (3.8), is indefinite. As is evident from the figures, all errors are $O(h^2)$ as was predicted by the theory of Fix, et al. [17]. The results of Figures 2 and 3 are representative of numerous computations involving the system (3.6) along with many different sets of boundary conditions. The $O(h^2)$ behavior of the errors was confirmed in every instance.

Our second example illustrates the versatility of the least squares finite element method. The problem considered is of the type discussed in section 2. The mean flow is assumed undirectional and of constant pressure. After appropriate nondimensionalizations we are considering the system (1.3) with

$$A_1 = \begin{pmatrix} \mu & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \mu}{\partial r} \\ 0 & 0 & 0 \end{pmatrix}.$$

In our specific example, we will take for μ the parabolic profile

$$\mu = 2(\tfrac{1}{4} - y^2)$$

Although the system (1.3) with the above values for A_1, A_2 and B looks rather simple it cannot be combined into a single second order equation for the pressure. Furthermore, an exact solution cannot be produced. Once again the region of interest will be the unit square and the boundary conditions are given by

$$p(0,y) = 1 \quad \text{and} \quad u(1,y) - p(1,y) = 0 \quad \text{for} \quad 0 < y < 1$$

and

$$v(x,0) + p(x,0) = 0 \quad \text{and} \quad v(x,1) - p(x,1) = 0 \quad \text{for} \quad 0 < x < 1 \ .$$

Figure 4 displays contour plots for approximations to the real and imaginary part of u, v, and p. The calculations were made for $k = \omega/c_0 = 7/4$.

This concludes our presentation of the least squares finite element method for the approximate solution of systems of the type (1.3). In summary, the principal virtues of the method are that it produces a symmetric, positive definite and sparse algebraic system for any value of the forcing frequency. It is applicable to general variable coefficient problems. In conjunction with the "criss-cross" triangles of Figure 1c the method yields the best posssible accuracy obtainable by elements of the approximating space. A final virtue not previously pointed out is that the treatment of boundary conditions via a boundary integral in the functional (3.1) enables one to easily implement complicated boundary conditions including those of the mixed type such as equation (2.8).

IV. The Finite Element Multigrid Method

In this section we describe the finite element multigrid method for second order equation of the type (1.4). We consider here only those problems for which equation (1.4) is elliptic, i.e. the matrix (a_{ij}) is positive definite. The multigrid method is not a method of discretization, but is a method of efficiently solving the algebraic system resulting from a given discretization. In conjunction with finite element methods, Nicolaides [12], [13], [14] has extensively analyzed the multigrid method. His theoretical and computational results apply to general real self-adjoint elliptic equations, including indefinite ones such as the Helmholtz equation. The basic result is that by use of the multigrid method the algebraic system resulting from a finite element discretization of an elliptic equation may be solved for within the discretization error in $O(N)$ operations, where N is the number of unknowns. In this work we apply the finite element multigrid method to non-self adjoint second order elliptic equations with complex valued coefficients. Our presentation

will be brief, and the reader is referred to the work of Nicolaides for any details which are not unique to the equations considered here.

In order to discretize equation (1.4), we start with a weak formulation of that equation and boundary conditions. To this end we let Γ_1 denote that part of the boundary Γ of the region Ω on which Dirichlet data is prescribed, i.e. equation (1.6) with $\underline{\beta} = \underline{0}$, and Γ_2 denote that part of Γ on which Neumann or mixed data is prescribed, i.e. equation (1.6) with $\underline{\beta} \neq \underline{0}$. Then the weak form of equation (1.4) is given by [19]:

Find a $P \in H_E^1(\Omega)$ such that

$$\int_\Omega \left\{ -\sum_{i,j=1}^{n} \left(a_{ij} \frac{\partial P}{\partial x_j} \frac{\partial \psi^*}{\partial x_i} \right) + \sum_{i=1}^{n} \left(b_j \frac{\partial P}{\partial x_j} \psi^* \right) + c P \psi^* \right\} + \int_{\Gamma_2} (G - \alpha P)\psi^* = \int_\Omega F\psi^* \tag{4.1}$$

for all $\psi \in H_0^1(\Omega)$. Here $H^r(\Omega)$ denotes the Sobolev space of order r, the spaces $H_0^1(\Omega)$ and $H_E^1(\Omega)$ are respectively defined by

$$H_0^1(\Omega) = \{\psi \in H^1(\Omega) \quad : \quad \psi = 0 \quad \text{on} \quad \Gamma_1\}$$

and

$$H_E^1(\Omega) = \{\psi \in H^1(\Omega) \quad : \quad \psi = g(x) \quad \text{on} \quad \Gamma_1\} .$$

We assume that $F \in H^0(\Omega)$ and $(\cdot)^*$ denotes the complex conjugate. We have also assumed that on Γ_2 the boundary condition (1.6) is of the usual flux type, i.e.

$$\beta_j = \sum_{i=1}^{n} a_{ij} n_i$$

where n_i is a component of the outer normal to Γ. Other boundary conditions can also be handled by making appropriate changes in the weak form (4.1) [19].

To find an approximate solution we first seek a $P^h \in P$ such that equation (4.1) holds for all $\psi^h \in \Psi$ where P and Ψ are finite dimensional subspaces of H_E^1 and H_0^1, respectively. The choice of these subspaces and their bases is accomplished in the standard manner [19] and will not be discussed here. In general P and Ψ will be identical except for boundary terms. Once the choices of P and Ψ and their bases are made, there results the matrix problem

$$K\underline{c} = \underline{d} \tag{4.2}$$

where the components of $\underline{c}$ are the coefficients in the representation of P^h in terms of the chosen basis for P. The matrix K will in general be complex valued, non-Hermitian and indefinite (in the sense that K plus its conjugate transpose is indefinite).

The implementation of the multigrid method for the solution of equation (4.2) exactly follows that described by Nicolaides [12], [13], [14] for real self adjoint indefinite problems, and therefore will not be discussed here. However, we do point out the crucial feature of that implementation which is responsible for the method's convergence for non-self adjoint indefinite problems. This feature is that on the coarsest grid, the algebraic problem is solved by a direct method, i.e. Gauss elimination. In general this coarse grid problem is small compared to the problem (4.2) and a direct solution does not incur an appreciable penalty insofar as computer time or storage is concerned. As pointed out by Nicolaides, some care must be exercised in choosing the coarsest grid since too coarse a grid will result in divergent iterates. In practice, this divergence can be monitored

by examining successive iterates, and if detected can be corrected by choosing a finer coarsest grid. Since a direct method is used on the coarsest grid, the multigrid method is not a pure iterative method. However, insofar as the advantages usually ascribed to iterative methods are concerned, e.g. low computer storage, the multigrid method certainly suffers little compared to the Gauss-Seidel or SOR methods. Furthermore the multigrid method requires $0(N)$ operations to satisfactorily solve (4.2) and in practice is found to converge much faster than other iterative methods (when these converge). The $0(N)$ operation count remains valid for indefinite non-self adjoint problems.

It is possible to use the Gauss-Seidel or SOR methods to solve for $\underline{c}$ if one uses the relation

$$K^*K\,\underline{c} = K^*\underline{d} \tag{4.3}$$

where K^* is the conjugate transpose of K. However, the matrix K^*K corresponds to a fourth order differential operator and the convergence of standard iterative methods for the solution of equation (4.3) is notoriously slow.

Many computational experiments have been carried out for equations of the type (1.4) subject to boundary conditions of the type (1.6). Details of the results of these computations can be found in Thomas [21]. Here, in Tables 1 and 2, we present a few of these results. Specifically, Table 1 contains information concerning the computational solution of the following problem:

$$
\left.
\begin{aligned}
P_{xx} + P_{yy} + CP &= 0 && \text{for } 0 \le x,y \le 1 \\
P(0,y) &= 0 && \text{for } 0 \le y \le 1 \\
P_x(1,y) + \beta P(1,y) &= \beta \cos(\sigma y) && \\
P_y(0,x) + \beta P(0,x) &= \beta \sin(\tfrac{\pi x}{2}) && \text{for } 0 \le x \le 1 \\
P_y(1,x) + \beta P(1,x) &= \sin(\tfrac{\pi x}{2})[\beta\cos\sigma - \sigma \sin\sigma] &&
\end{aligned}
\right\} \quad (4.4)
$$

whose exact solution is

$$P(x,y) = \sin(\tfrac{\pi}{2}x)\ \cos(\sigma y). \qquad (4.5)$$

Here $\sigma = [c-(\pi/2)^2]^{\frac{1}{2}}$. The parameters c and β given in the table refer to the parameters appearing in the description of the problem. The parameter M_0 and M_1 refer to the number of intervals in each coordinate direction for the coarsest and finest grids, respectively, used in the computation. Finally, the last column in the table gives the computed "effective spectral radius" (ESR) for an average multigrid cycle, where ESR is defined by

$$\text{ESR} = \left(\frac{L_2 \text{ error after j multigrid cycles}}{\text{Initial } L_2 \text{ error}} \right)^{\frac{1}{j}}$$

The computations displayed were performed using $j = 5$, although variations in j do not appreciably affect the results.

Lines (i) and (ii) of Table 1 illustrate the result that the convergence rate of the multigrid method is unaffected by the size of the finest grid, i.e. the number of unknowns N. Lines (iii) - (v) are indicative of the divergence of the method for not fine enough coarse grids. This divergence is largely unaffected by the size of the finest grid, as can be

seen by examining lines (iii) and (iv). However, if the coarsest grid is refined as in line (v), convergence is attained. Finally, the table indicates that the ESR is bounded away from unity, i.e. is not $1-N^{-p}$, where $p > 0$. This result, together with the independence of the ESR from N, is the cause of the convergence of the method in $O(N)$ operation.

Table 2 contains information concerning the computational solution of the following non-self-adjoint problem:

$$\left.\begin{aligned} &P_{xx} + P_{yy} + P_x + P_y + cP = \frac{\pi}{2}\cos\frac{\pi x}{2}\cos(\sigma y) - \sigma \sin\frac{\pi x}{2}\sin(\sigma y) \quad \text{for } 0 \le x,y \le 1 \\ &P(0,y) = 0 \\ &P(1,y) = \cos(\sigma y) \qquad \text{for } 0 \le y \le 1 \\ &P(x,0) = \sin\left(\frac{\pi x}{2}\right) \\ &P(x,1) = \cos(\sigma)\sin\left(\frac{\pi x}{2}\right) \qquad \text{for } 0 \le x \le 1 \end{aligned}\right\} \tag{4.6}$$

whose exact solution is again equation (4.5). The table again indicates that the ESR is bounded away from unity. Furthermore, it indicates that some improvement can be gained by making the coarsest grid coarser.

We close by noting that some ideas concerning multigrid methods for finite difference discretization of indefinite problems have been advanced by Brandt [22].

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APPENDIX I: Remarks Concering Radiation Conditions

In this appendix we collect some remarks concerning radiation conditions for exterior problems. We assume a familiarity with the theory of radiation conditions for the Helmholtz equation (1.7).

Problems with Flow at Infinity

We consider problems for which at large distances (in all directions) from the origin, the fluid media is in a state of uniform flow. For instance, consider the scattering of sound by a body moving with uniform subsonic velocity through a fluid. Then, relative to an observer fixed with the body, the fluid at infinity moves with a uniform velocity. In general the propagation of sound will be governed by the system (1.3) with coefficient given by equation (2.4). However, in the far field, this system can be approximated by the system (2.6) if we align the x-axis with the direction of the flow at infinity. As was noted previously, the system (2.6) may be combined into the single second order equation

$$\Delta P - M^2 P_{xx} - 2ikMP_x + k^2 P = 0 \qquad \text{(A1.1)}$$

where Δ is the Laplacian operator. The transformation

$$P = \sigma \exp\{ikMx/(1-M^2)\} \quad \text{and} \quad \xi = x/(1-M^2)^{\frac{1}{2}} \qquad \text{(A1.2)}$$

reduce (A1.1) to the Helmholtz equation

$$\Delta_\xi \sigma + \hat{k}^2 \sigma = 0 \qquad \text{(A1.3)}$$

where Δ_ξ is the Laplacian operator in (ξ,y) coordinates and $\hat{k} = k/(1-M^2)^{\frac{1}{2}}$. For three dimensional problems the same construction is possible, resulting in equation (A1.3) with Δ_ξ representing the Laplacian operator in (ξ,y,z) coordinate. For equation (A1.3) the appropriate radiation conditions is given by [9]

$$\lim_{\eta\to\infty}\left\{\eta^{\frac{n-1}{2}}\left|\frac{\partial\sigma}{\partial\eta} + i\hat{k}\sigma\right|\right\} = 0 \qquad \text{(A1.4)}$$

where n is the number of space dimensions and $\eta = (\xi^2+y^2)^{\frac{1}{2}}\ [(\xi^2+y^2+z^2)^{\frac{1}{2}}]$ in two [three] dimensions. Substitution of the transformation (A1.2) and (A1.4) yields for $n = 3$

$$\lim_{r\to\infty}\left\{r\left|\frac{\partial P}{\partial r} + i\hat{k}\left(1 - \frac{xM}{r\sqrt{1-M^2}}\right)P\right|\right\} = 0 \qquad \text{(A1.5)}$$

where $r = (x^2+y^2+z^2)^{\frac{1}{2}}$. Equation (A1.5) is then the appropriate radiation condition for equation (A1.1) in three dimensions. The analogous two dimensional formula may be derived in a similar manner.

In computations the radiation condition (A1.5) [or A1.4] may be imposed by evaluating the limit at a finite radius, or more generally, at a bounded surface in space. This introduces an error of $O(1/R^2)$ in the computations, where R is the distance at which the condition is imposed. This error should be balanced with other discretization errors appearing in the computations. This can be accomplished by choosing R to be sufficiently large.

The Effect of Approximate Radiation Conditions on Uniqueness

The radiation condition (1.8) renders unique the solution of the Helmholtz equation (1.7) in exterior domains [9,10]. If the radiation

condition is applied at a finite distance, the solution of equation (1.7) reamins unique. This can be illustrated by the following simple example

$$\left\{\begin{array}{rl} \phi_{xx} + k^2\phi = 0 & 0 < x < 1 \\ \phi(0) = 0 & \\ \phi_x(1) + ik\,\phi(1) = 0 & \end{array}\right\} \tag{A1.6}$$

The solution of the differential equation and the first boundary condition is given by

$$\phi = \alpha \sin(kx) \tag{A1.7}$$

where α is a constant. Then applying the "radiation condition" at $x = 1$ results in

$$\alpha k \exp\{ik\} = 0 \tag{A1.8}$$

which is satisfied only if α or k vanish. In either case, the solution (A1.7) is the trivial solution. Therefore the solution of an inhomogeneous version of the problem (A1.6) will be unique.

Problems similar to (A1.6) can be produced for two and three dimensions. For instance, for $n = 3$ consider

$$\left\{\begin{array}{rl} \phi_{rr} + \frac{2}{r}\phi_r + k^2\phi = 0 & 1 < r < R \\ \phi(1) = 0 & \\ R[\phi_r(R) + ik\,\phi(R)] = 0 & \end{array}\right\} \tag{A1.9}$$

Instead of equation (A1.8) we are led to

$$\alpha\, k \exp[i k (R-1)] - \frac{1}{R} \sin[k(R-1)] = 0$$

which for $R > 1$ and k real implies that $\alpha = 0$. Therefore the solution of (A1.9) vanishes.

Consistent numerical approximation of the Helmholtz equation with the radiation condition imposed at a finite boundary will also be unique. To illustrate this, let us consider a central difference approximation to the problem (A1.6), i.e.

$$\left\{ \begin{array}{c} \phi_0 = 0 \\ \phi_{j-1} - 2\phi_j + \phi_{j+1} + k^2h^2\phi_j = 0 \quad \text{for} \quad j = 1,\ldots J-1 \\ \frac{1}{h}(\phi_J - \phi_{J-1}) + i\frac{k}{2}[\phi_J + \phi_{J-1}] = 0 \end{array} \right\} \qquad \text{(A1.10)}$$

where $h = 1/J$ and ϕ_j approximates $\phi(jh)$. The solution of the first two equations in (A1.10) is given by

$$\phi_j = \beta \sin(j\, k\, h). \qquad \text{(A1.11)}$$

Substitution into the third equation in (A1.10) yields

$$k\,\beta\left[\exp\{ik\}(1 - \frac{ikh}{2}) + 0(k^2h^2)\right] = 0$$

For h sufficiently small this implies that k or β vanish so that the solution (A1.11) also vanishes. Then solution of inhomogeneous versions of problem (A1.10) will be unique.

	c	β	M_0	M_1	ESR
i	5	1	2	8	.765
ii	5	1	2	16	.761
iii	5	i	2	8	2.127
iv	5	i	2	16	2.158
v	5	i	8	16	.739
vi	5	1 + i	8	16	.764
vii	15	1	8	16	.787
viii	15	i	8	16	.737
ix	15	1 + i	8	16	.801

Table 1: Multigrid computations for problem (4.4).

c	M_0	M_1	ESR
1	4	16	.664
3	4	16	.608
5	4	16	.526
5	8	16	.546
12	4	16	.693
12	8	16	.721

Table 2: Multigrid computation for problem (4.6).

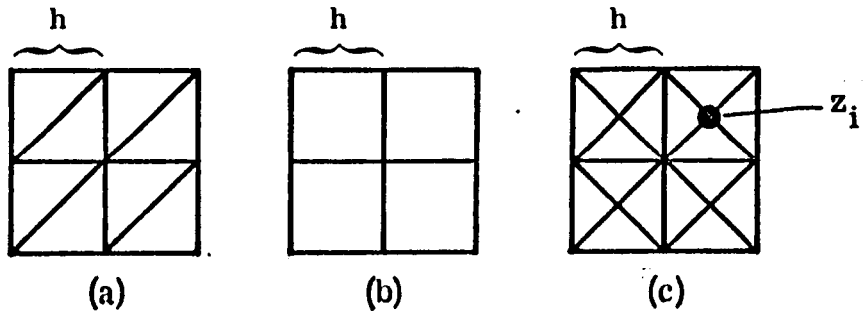

FIGURE 1: GRIDS. A) DIRECTIONAL; B) RECTANGULAR; C) CRISS-CROSS.

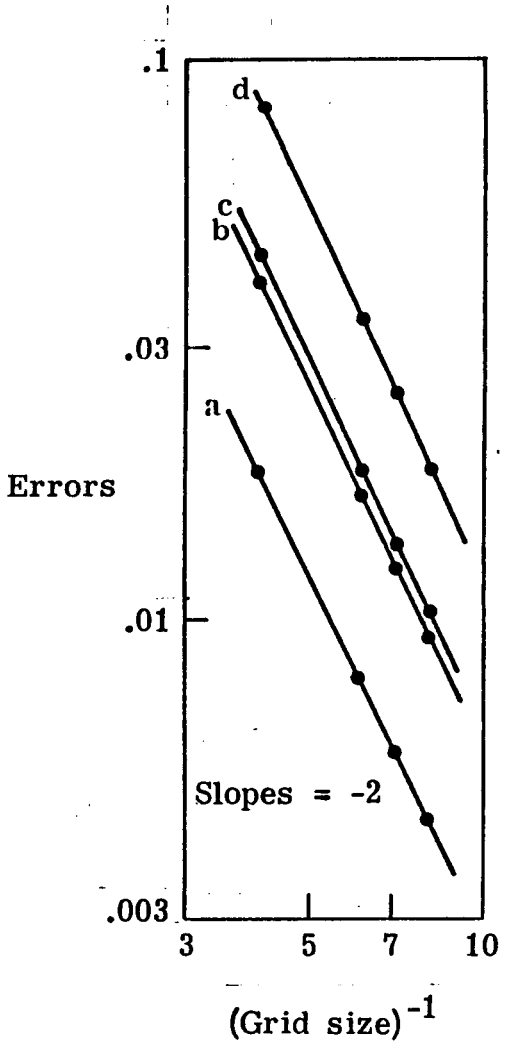

Figure 2: Errors for problem (3.6) with k = 1.
a) e_p; b) e_u; c) e_v; d) E_p.

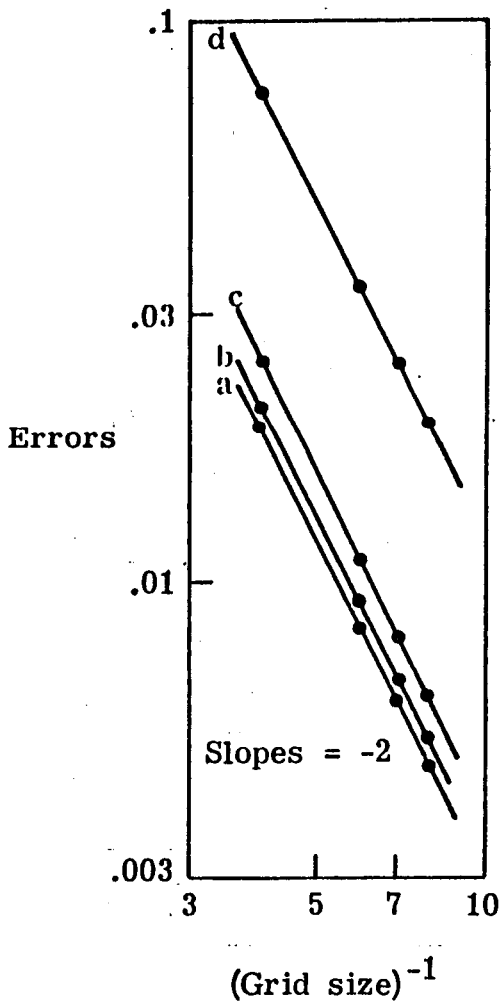

Figure 3: Errors for problem (3.6) with $k = 7/4$.
a) e_p; b) e_u; c) e_v; d) E_p.

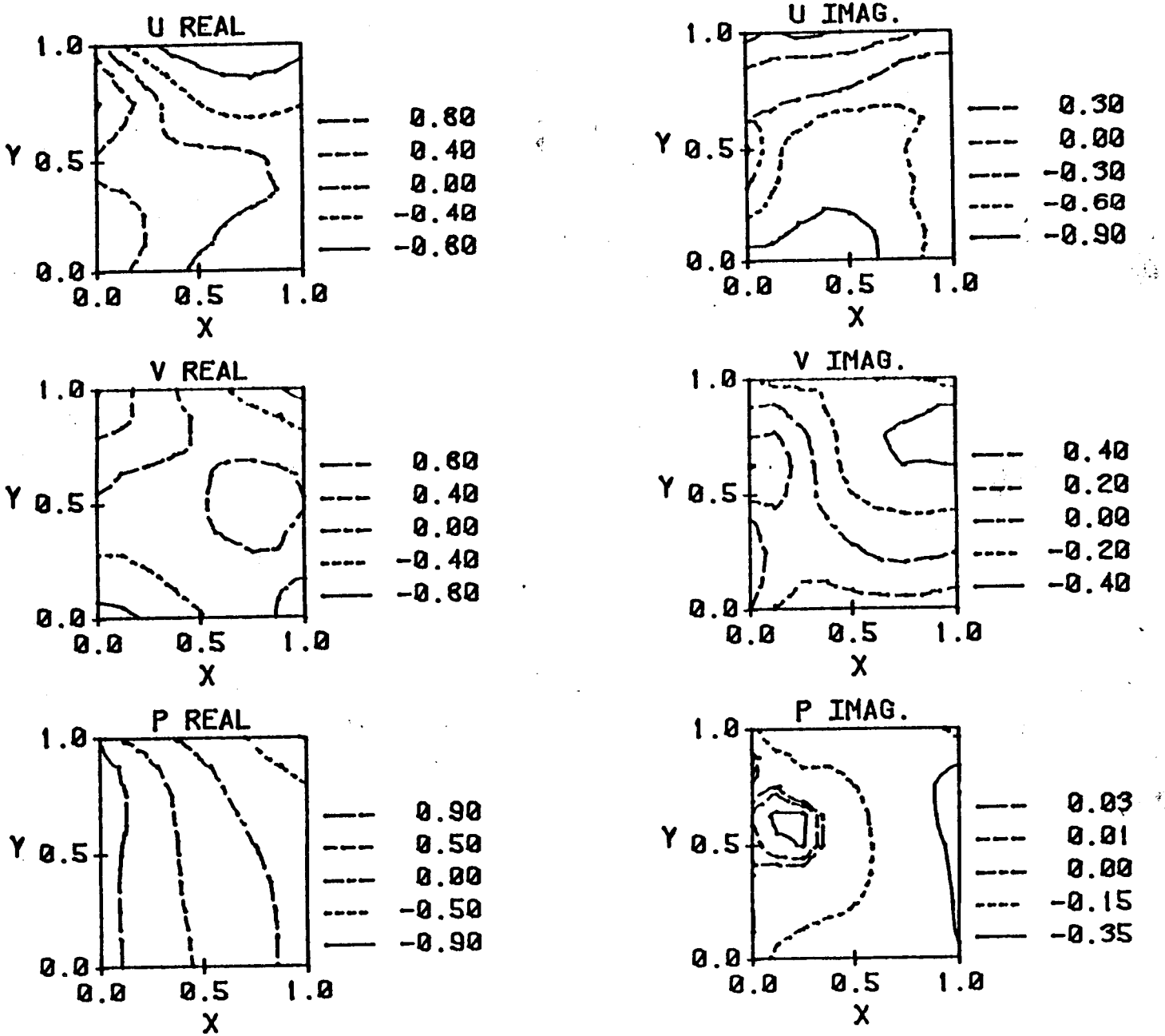

FIGURE 4: COMPUTED SOLUTION FOR VARIABLE COEFFICIENT PROBLEM.